Energy-momentum tensor for a Casimir apparatus in a weak gravitational field

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Abstract

The influence of the gravity acceleration on the regularized energy-momentum tensor of the quantized electromagnetic field between two plane parallel conducting plates is derived. We use Fermi coordinates and work to first order in the constant acceleration parameter. A perturbative expansion, to this order, of the Green functions involved and of the energy-momentum tensor is derived by means of the covariant geodesic point splitting procedure. In correspondence to the Green functions satisfying mixed and gauge-invariant boundary conditions, and Ward identities, the energy-momentum tensor is covariantly conserved and satisfies the expected relation between gauge-breaking and ghost parts, while a new simple formula for the trace anomaly is obtained to first order in the constant acceleration. A more systematic derivation is therefore obtained of the theoretical prediction according to which the Casimir device in a weak gravitational field will experience a tiny push in the upwards direction.

I. INTRODUCTION

The use of Green-function methods in curved spacetime, and the theoretical analysis of the Casimir effect, are two relevant branches of modern quantum field theory. In the former case, some progress along the years can be outlined as follows.

(i) In Ref. [1], the covariant geodesic point separation is applied to evaluate the vacuum expectation value of the energy-momentum tensor $T_{\mu\nu}$ for a massive scalar field in an arbitrary gravitational field. In Ref. [2], the same author performs regularization, renormalization and covariant geodesic point separation for spin $0, 1/2$ and 1 fields, massive or massless, on an arbitrary curved background; he also finds which terms in the vacuum expectation value of $T_{\mu\nu}$ vanish within this framework.

(ii) In Ref. [3], the authors obtain a momentum-space representation of the Feynman propagator $G(x, x')$ for scalar and spin-$\frac{1}{2}$ fields propagating in arbitrary curved spacetimes. Their construction uses Riemann normal coordinates with origin at the point x' and is therefore only valid for points x lying in a normal neighbourhood of x'. The resulting momentum-space representation is equivalent to the Schwinger–DeWitt proper-time representation.

(iii) In Ref. [4], electromagnetic and scalar fields are quantized in the region near an arbitrary smooth boundary. The authors find that the components of $\langle T_{\mu\nu}(x)\rangle_{\rm ren}$ generically diverge in a nonintegrable manner as x approaches the boundary of the manifold from the interior. They therefore conclude that perfect conductor boundary conditions are pathological, in that the distribution of energy-momentum that they would entail if they were actually to obtain for arbitrarily high frequencies and short wavelengths would be such as to produce an infinite *physically observable* gravitational field.

(iv) In Ref. [5], the authors derive the symmetric Hadamard representation for scalar and photon Green functions, and use these representations to give a simple definition for their associated renormalized energy-momentum tensors.

(v) In Ref. [6], the full asymptotic expansion of the Feynman photon Green function at small values of the world function, as well as its explicit dependence on the gauge parameter, are obtained without adding by hand a mass term to the DeWitt–Faddeev–Popov Lagrangian. Coincidence limits of second covariant derivatives of the associated Hadamard function are also evaluated.

On the other hand, an important property of quantum electrodynamics is that suitable differences of zero-point energies of the quantized electromagnetic field can be made finite and produce measurable effects such as the tiny attractive force among perfectly conducting parallel plates known as the Casimir effect [7]. This is a remarkable quantum mechanical effect that makes itself manifest on a macroscopic scale. For perfect reflectors and metals the Casimir force can be attractive or repulsive, depending on the geometry of the cavity, whereas for dielectrics in the weak-reflector approximation it is always attractive, independently of the geometry [8]. The Casimir effect can be studied within the framework of boundary effects in quantum field theory, combined with zeta-function regularization or Green-function methods, or in more physical terms, i.e. on considering van der Waals forces [9] or scattering problems [10]. Casimir energies are also relevant in the attempt of building a quantum theory of gravity and of the universe [11].

For these reasons, in Ref. [12] we evaluated the force produced by a weak gravitational field on a rigid Casimir cavity. Interestingly, the resulting force was found to have opposite direction with respect to the gravitational acceleration; moreover, we found that the current experimental sensitivity of small force macroscopic detectors would make it possible, at least in principle, to measure such an effect [12]. More precisely, the gravitational force on the Casimir cavity might be measured provided one were able to use rigid cavities and find an efficient force modulation method [12]. Rigid cavities, composed by metal layers separated by a dielectric layer, make it possible to reach separations as small as $5 \div 10$ nm and allow to build multi-cavity structures, made by a sequence of such alternate layers. If an efficient modulation method could be found, it would be possible to achieve a modulated force of order 10^{-14} N in the earth's gravitational field. The measure of such a force, already possible with current small-force detectors on macroscopic bodies, might open the way to the first test of the gravitational field influence on vacuum energy [12]. In Ref. [12], calculations were based on simple assumptions and the result can be viewed as a reasonable "*first order*" generalization of $T_{\mu\nu}$ from Minkowski to curved space-time. The present paper is devoted to a deeper understanding and to more systematic calculations of the interaction of a weak gravitational field with a Casimir cavity. To first order in our approximation the former value of the force exerted by the field on the cavity is recovered. But here we also find a trace anomaly for the energy-momentum tensor.

We consider a plane-parallel Casimir cavity, made of ideal metallic plates, at rest in the gravitational field of the earth, with its plates lying in a horizontal plane. We evaluate the influence of the gravity acceleration g on the Casimir cavity but neglect any variation of the gravity acceleration across the cavity, and therefore we do not consider the influence of tidal forces. The separation a between the plates is taken to be much smaller than the extension of the plates, so that edge effects can be neglected. We obtain a perturbative expansion of the energy-momentum tensor of the electromagnetic field inside the cavity, in terms of the small parameter $\epsilon \equiv 2ga/c^2$, to first order in ϵ. For this purpose, we use a Fermi [13, 14] coordinates system (t, x, y, z) rigidly connected to the cavity. The construction of these coordinates involves only invariant quantities such as the observer's proper time, geodesic distances from the world-line, and components of tensors with respect to a tetrad [14]. This feature makes it possible to obtain a clear identification of the various terms occurring in the metric. In our analysis we adopt the covariant point-splitting procedure [1, 15] to compute the perturbative expansion of the relevant Green functions. Gauge invariance plays a crucial role and we check it up to first order by means of the Ward identity. We also evaluate the Casimir energy and pressure, and in this way we obtain a sound derivation of the result in Ref. [12], according to which the Casimir device in a weak gravitational field will experience a tiny push in the upwards direction. Use is here made of mixed boundary conditions on the potential plus Dirichlet conditions on ghost fields. Concluding remarks are presented in Sec. VI, while relevant details are given in the Appendices.

II. SPIN-1 AND SPACETIME FORMALISM

The classical action functional for the Maxwell potential $A_\mu(x)$ reads

$$S[A_\mu] = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\sqrt{-g}d^4x, \tag{2.1}$$

where the field strength $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu = A_{\nu;\mu} - A_{\mu;\nu}$. By virtue of the gauge invariance of the action, the differential operator $\frac{\delta^2 S}{\delta A_\mu \delta A_{\nu'}}$ is singular (see Appendix A). To take care of this problem one should add a gauge-breaking term, which leads to a nonsingular wave operator $\widehat{U}^{\alpha\beta'}$ on the potential. In the Lorenz (this is not Lorentz!) gauge [16], and with the Feynman choice for the gauge parameter (see Appendix A), this is $-\frac{1}{2}(\nabla^\mu A_\mu)^2$. Last, a ghost term $\chi^{;\alpha}\psi_{;\alpha}$ is necessary, where χ and ψ are independent ghost fields [17]. The full action is therefore

$$S[A_\mu, \chi, \psi] = \int \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\nabla^\mu A_\mu)^2 + \chi^{;\alpha} \psi_{;\alpha} \right] \sqrt{-g} d^4x, \tag{2.2}$$

with field equations [2]

$$\int \widehat{U}^{\alpha\beta'} A_{\beta'} d^4x' = 0, \tag{2.3}$$

$$\int \widehat{U}(x, x') \psi(x') d^4x' = 0, \tag{2.4}$$

having defined (here $\delta^{\alpha\beta'} \equiv g^{\alpha\beta} \delta(x, x')$)

$$\widehat{U}^{\alpha\beta'} \equiv \frac{\delta^2 S}{\delta A_\alpha \delta A_{\beta'}} = \sqrt{-g} \left(\delta^{\alpha\beta'}{}_{;\rho}{}^{\rho} - R^\alpha_\rho \delta^{\rho\beta'} \right), \tag{2.5}$$

$$\widehat{U}(x, x') \equiv \frac{\delta^2 S}{\delta\chi \delta\psi'} = -\sqrt{-g} \delta_{;\rho}{}^{\rho}(x, x'). \tag{2.6}$$

The energy-momentum tensor is obtained from functional differentiation of the full action (2.2) according to

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}, \tag{2.7}$$

and turns out to be the sum of three contributions $T_A, T_B, T_{\rm gh}$ resulting from Maxwell action, gauge-breaking term and ghost action, respectively [2] (no mass term occurs since we do not add by hand any mass for photons or ghost fields, unlike Ref. [2]). This leads eventually to the following vacuum expectation value:

$$\langle T^{\mu\nu} \rangle = \langle T_A^{\mu\nu} \rangle + \langle T_B^{\mu\nu} \rangle + \langle T_{\rm gh}^{\mu\nu} \rangle, \tag{2.8}$$

where, on defining the photon and ghost Hadamard functions, respectively, as

$$H_{\mu\nu}(x, x') \equiv \langle [A_\mu(x), A_\nu(x')]_+ \rangle \equiv H_{\mu\nu'}, \tag{2.9}$$

$$H(x, x') \equiv \langle [\chi(x), \psi(x')]_+ \rangle, \tag{2.10}$$

jointly with

$$\begin{aligned} \langle F_{\rho\alpha} F_{\tau\beta} \rangle \; &= \; \lim_{x' \to x} \frac{1}{4} \left[H_{\alpha\beta';\rho\tau'} + H_{\beta\alpha';\tau\rho'} - H_{\alpha\tau';\rho\beta'} - H_{\tau\alpha';\beta\rho'} - H_{\rho\beta';\alpha\tau'} \right. \\ &\left. - H_{\beta\rho';\tau\alpha'} + H_{\rho\tau';\alpha\beta'} + H_{\tau\rho';\beta\alpha'} \right], \end{aligned} \tag{2.11}$$

one has

$$\langle T_A^{\mu\nu} \rangle = \lim_{x' \to x} \left[-\frac{1}{4} \left(g^{\mu\rho} g^{\nu\tau} - \frac{1}{4} g^{\mu\nu} g^{\tau\rho} \right) g^{\alpha\beta} \langle F_{\rho\alpha} F_{\tau\beta} \rangle \right], \tag{2.12}$$

$$\langle T_B^{\mu\nu}\rangle = \lim_{x'\to x}\left[-\frac{1}{4}g^{\alpha\beta}\left(g^{\mu\rho}g^{\nu\tau}+g^{\mu\tau}g^{\nu\rho}-g^{\mu\nu}g^{\tau\rho}\right)\left(H_{\beta\tau';\alpha\rho}+H_{\tau\beta';\rho\alpha'}\right)\right], \tag{2.13}$$

$$\langle T_{\rm gh}^{\mu\nu}\rangle = \lim_{x'\to x}\left[-\frac{1}{4}g^{\alpha\beta}\left(g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}-g^{\mu\nu}g^{\alpha\beta}\right)\left(H_{;\alpha\beta'}+H_{;\beta\alpha'}\right)\right]. \tag{2.14}$$

Following Christensen [2], we separate points symmetrically, i.e.

$$[A_{\alpha;\rho},A_{\beta;\tau}]_+ = \lim_{x'\to x}\frac{1}{2}\left\{[A_{\alpha';\rho'},A_{\beta;\tau}]_+ + [A_{\alpha;\rho},A_{\beta';\tau'}]_+\right\}. \tag{2.15}$$

In the applications considered in Sec. IV, the coincidence limit will be taken in two separate steps: first with respect to time and the coordinates spanning the plates, and eventually with respect to the z-coordinate (see Appendix B for details).

In the implementation of Eq. (2.15) one needs the geodesic parallel displacement bivector $g^{\mu}_{\ \nu'}$ (in general, bi-tensors behave as a tensor both at x and at x') which effects parallel displacement of vectors along the geodesic from x' to x. In general, it is defined by the differential equations

$$\sigma^{;\rho}g^{\mu}_{\ \nu';\rho} = \sigma^{;\tau'}g^{\mu}_{\ \nu';\tau'} = 0, \tag{2.16}$$

$\sigma(x,x')$ being the Ruse–Synge world function [18], equal to half the geodesic distance between x and x', jointly with the coincidence limit (boundary condition)

$$\lim_{x'\to x} g^{\mu}_{\ \nu'} \equiv [g^{\mu}_{\ \nu'}] = \delta^{\mu}_{\ \nu}. \tag{2.17}$$

The bivector $g^{\mu}_{\ \nu'}$, when acting on a vector $B^{\nu'}$ at x', gives the vector $\overline{B}^{\mu}$, which is obtained by parallel transport of $B^{\nu'}$ to x along the geodesic connecting x and x', i.e.

$$\overline{B}^{\mu} = g^{\mu}_{\ \nu'}B^{\nu'}. \tag{2.18}$$

For the reasons described in the Introduction, we use Fermi coordinates. With our choice, the z-axis coincides with the vertical upwards direction, while the (x,y) coordinates span the plates, whose equations are $z=0$ and $z=a$, respectively. The resulting line element for a nonrotating system is therefore [13]

$$ds^2 = -c^2\left(1+\epsilon\frac{z}{a}\right)dt^2 + dx^2 + dy^2 + dz^2 + \mathrm{O}(|\ x\ |^2) = \eta_{\mu\nu}dx^\mu dx^\nu - \epsilon\frac{z}{a}c^2dt^2, \tag{2.19}$$

where the perturbation parameter $\epsilon \equiv 2ga/c^2$, while $\eta_{\mu\nu}$ is the flat Minkowski metric $\mathrm{diag}(-1,1,1,1)$.

III. GREEN FUNCTIONS

For any field theory, once that the invertible differential operator U_{ij} in the functional integral is given (see Eq. (A4) in Appendix A), the corresponding Green functions satisfy the condition

$$U_{ij}G^{jk} = -\delta_i^{\ k}, \tag{3.1}$$

and are boundary values of holomorphic functions. The choice of boundary conditions will determine whether we deal with advanced Green functions G^{+jk}, for which the integration contour passes below the poles of the integrand on the real axis, or retarded Green functions G^{-jk}, for which the contour passes instead above all poles on the real axis, or yet other types of Green functions. Among these, a key role is played by the Feynman Green function G_F^{jk}, obtained by choosing a contour that passes below the poles of the integrand that lie on the negative real axis and above the poles on the positive real axis. If one further defines the Green function [17]

$$\overline{G}^{jk} \equiv \frac{1}{2}(G^{+jk} + G^{-jk}), \tag{3.2}$$

one finds in stationary backgrounds (for which the metric is independent of the time coordinate, so that there exists a timelike Killing vector field) that the Feynman Green function has a real part equal to $\overline{G}^{jk}$, and an imaginary part equal to the Hadamard function H^{jk}, i.e.

$$H^{jk}(x,x') \equiv -2i\Big[G_F^{jk}(x,x') - \overline{G}^{jk}(x,x')\Big]. \tag{3.3}$$

This relation can be retained as a definition when the background is nonstationary; in such a case, however, $H^{jk}(x,x')$ is generally no longer real [17].

In particular, the photon Green function $G_{\lambda\nu'}$ in a curved spacetime with metric $g_{\mu\nu}$ solves the equation [6]

$$\sqrt{-g}P_\mu^{\ \lambda}(x)G_{\lambda\nu'} = g_{\mu\nu}\delta(x,x'). \tag{3.4}$$

The wave operator $P_\mu^{\ \lambda}$ results from the gauge-fixed action (2.2) with Lorenz gauge-fixing functional $\Phi_L(A) \equiv \nabla^\mu A_\mu$, and having set to 1 the gauge parameter of the general theory, so that (cf. Eq. (2.5))

$$P_\mu^{\ \lambda}(x) = -\delta_\mu^{\ \lambda}\Box_x + R_\mu^{\ \lambda}(x), \tag{3.5}$$

where $\Box_x \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta(x)$. Since we need the action of the gauge-field operator $P_\mu^{\ \lambda}(x)$ on the photon Green functions, it is worth noticing that

$$D_{\beta\lambda\nu'} \equiv \nabla_\beta G_{\lambda\nu'} = \partial_\beta G_{\lambda\nu'} - \Gamma^\mu_{\beta\lambda} G_{\mu\nu'}, \tag{3.6}$$

$$Q_{\alpha\beta\lambda\nu'} \equiv \nabla_\alpha \nabla_\beta G_{\lambda\nu'} = \nabla_\alpha D_{\beta\lambda\nu'} = \partial_\alpha D_{\beta\lambda\nu'} - \Gamma^\mu_{\alpha\beta} D_{\mu\lambda\nu'} - \Gamma^\mu_{\alpha\lambda} D_{\beta\mu\nu'}. \tag{3.7}$$

The Christoffel coefficients for our metric (2.19) read

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta} \right) = -\frac{1}{2}\frac{\epsilon}{a} \left(\eta^{\alpha 0} \delta^0_\beta \delta^3_\gamma + \eta^{\alpha 0} \delta^0_\gamma \delta^3_\beta - \eta^{3\alpha} \delta^0_\gamma \delta^0_\beta \right). \tag{3.8}$$

Since the connection coefficients, to first order in ϵ, are constant, we realize that the Ricci curvature tensor vanishes to this order. On expanding (this is, in general, only an asymptotic expansion)

$$G_{\lambda\nu'} \sim G^{(0)}_{\lambda\nu'} + \epsilon \, G^{(1)}_{\lambda\nu'} + \mathrm{O}(\epsilon^2), \tag{3.9}$$

we get

$$D_{\beta\lambda\nu'} = \partial_\beta G_{\lambda\nu'} - \Gamma^\mu_{\beta\lambda} G_{\mu\nu'} = \partial_\beta G_{\lambda\nu'} - \Gamma^\mu_{\beta\lambda} G^{(0)}_{\mu\nu'}, \tag{3.10}$$

so that

$$Q_{\alpha\beta\lambda\nu'} = \partial_\alpha \partial_\beta G_{\lambda\nu'} - \partial_\alpha \left[\Gamma^\mu_{\beta\lambda} G^{(0)}_{\mu\nu'} \right] - \Gamma^\mu_{\alpha\beta} \partial_\mu G^{(0)}_{\lambda\nu'} - \Gamma^\mu_{\alpha\lambda} \partial_\beta G^{(0)}_{\mu\nu'}, \tag{3.11}$$

and finally

$$\begin{aligned} \Box_x G_{\lambda\nu'} &= g^{\alpha\beta} \nabla_\alpha \nabla_\beta G_{\lambda\nu'} = \left(\eta^{\alpha\beta} + \epsilon \frac{z}{a} \delta^\alpha_0 \delta^\beta_0 \right) \nabla_\alpha \nabla_\beta \left[G^{(0)}_{\lambda\nu'} + \epsilon G^{(1)}_{\lambda\nu'} \right] \\ &= \eta^{\alpha\beta} \Big[\partial_\alpha \partial_\beta G^{(0)}_{\lambda\nu'} + \epsilon \partial_\alpha \partial_\beta G^{(1)}_{\lambda\nu'} - \Gamma^\mu_{\beta\lambda} G^{(0)}_{\mu\nu',\alpha} \\ &\quad - \Gamma^\mu_{\alpha\beta} G^{(0)}_{\lambda\nu',\mu} - \Gamma^\mu_{\alpha\lambda} G^{(0)}_{\mu\nu',\beta} \Big] - \epsilon \frac{z}{a} \delta^\alpha_0 \delta^\beta_0 \partial_\alpha \partial_\beta G^{(0)}_{\lambda\nu'}. \end{aligned} \tag{3.12}$$

We therefore get, to first order in ϵ,

$$\Box^0 G^{(0)}_{\mu\nu'} = J^{(0)}_{\mu\nu'}, \tag{3.13}$$

$$\Box^0 G^{(1)}_{\mu\nu'} = J^{(1)}_{\mu\nu'}, \tag{3.14}$$

where

$$J^{(0)}_{\mu\nu'} \equiv -\eta_{\mu\nu} \delta(x, x'), \tag{3.15}$$

$$\epsilon J^{(1)}_{\mu\nu'} \equiv \frac{z}{a} \epsilon \left(\frac{\eta_{\mu\nu}}{2} + \delta^0_\mu \delta^0_\nu \right) \delta(x, x') + 2\eta^{\rho\sigma} \Gamma^\tau_{\sigma\mu} G^{(0)}_{\tau\nu',\rho} + \eta^{\rho\sigma} \Gamma^\tau_{\rho\sigma} G^{(0)}_{\mu\nu',\tau} - \frac{z}{a} \epsilon G^{(0)}_{\mu\nu',00}, \tag{3.16}$$

with $\Box^0 \equiv \eta^{\alpha\beta} \partial_\alpha \partial_\beta = -\partial_0^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$.

To fix the boundary conditions we note that, on denoting by $\vec{E}_t$ and $\vec{H}_n$ the tangential and normal components of the electric and magnetic fields, respectively, a sufficient condition to obtain

$$\vec{E}_t\Big|_S = 0, \quad \vec{H}_n\Big|_S = 0, \tag{3.17}$$

on the boundary S of the device, is to impose Dirichlet boundary conditions on $A_0(\vec{x}), A_1(\vec{x}), A_2(\vec{x})$ [19] at the boundary $z = 0$, $z = a$. The boundary condition on A_3 is determined by requiring that the gauge-fixing functional, here chosen to be of the Lorenz type, should vanish on the boundary (see Appendix A). This implies

$$A^{\mu}_{;\mu}\big|_S = 0 \Rightarrow A^3_{;3}\big|_S = (g^{33}\partial_3 A_3 - g^{\mu\nu}\Gamma^3_{\mu\nu}A_3)\big|_S = 0 \ . \tag{3.18}$$

To first order in ϵ, these conditions imply the following equations for Green functions:

$$G^{(0)}_{\mu\nu'}\Big|_S = 0, \quad \mu = 0, 1, 2, \ \forall \nu', \tag{3.19}$$

$$\partial_3 G^{(0)}_{3\nu'}\Big|_S = 0, \quad \forall \ \nu', \tag{3.20}$$

$$G^{(1)}_{\mu\nu'}\Big|_S = 0, \quad \mu = 0, 1, 2, \ \forall \nu', \tag{3.21}$$

$$\partial_3 G^{(1)}_{3\nu'}\Big|_S = -\frac{1}{2a}\, G^{(0)}_{3\nu'}\Big|_S, \quad \forall \ \nu', \tag{3.22}$$

hence we find that the third component of the potential A_μ satisfies homogeneous Neumann boundary conditions to zeroth order in ϵ and inhomogeneous boundary conditions to first order.

Since $J^{(0)}_{\mu\nu'}$ is diagonal, by virtue of the homogeneity of the boundary conditions, $G^{(0)}_{\lambda\nu'}$ turns out to be diagonal. On the contrary, $J^{(1)}_{\mu\nu'}$ has two off-diagonal contributions: $J^{(1)}_{03}$ and $J^{(1)}_{30}$, so that $G^{(1)}_{\mu\nu'}$ is nondiagonal. Let us write down explicitly the expressions for the various components of $J^{(1)}_{\lambda\nu'}$, i.e.

$$aJ^{(1)}_{00'} = \frac{z}{2}\delta(x, x') - zG^{(0)}_{00',00} + \frac{1}{2}G^{(0)}_{00',3}, \tag{3.23}$$

$$aJ^{(1)}_{03'} = -G^{(0)}_{33',0}, \tag{3.24}$$

$$aJ^{(1)}_{11'} = \frac{z}{2}\delta(x, x') - zG^{(0)}_{11',00} - \frac{1}{2}G^{(0)}_{11',3}, \tag{3.25}$$

$$aJ^{(1)}_{22'} = \frac{z}{2}\delta(x, x') - zG^{(0)}_{22',00} - \frac{1}{2}G^{(0)}_{22',3}, \tag{3.26}$$

$$aJ^{(1)}_{33'} = \frac{z}{2}\delta(x, x') - zG^{(0)}_{33',00} - \frac{1}{2}G^{(0)}_{33',3}, \tag{3.27}$$

$$aJ^{(1)}_{30'} = -G^{(0)}_{00',0}. \tag{3.28}$$

Now we are in a position to evaluate, at least formally (see below), the solutions to zeroth and first order, and we get

$$G^{(0)}_{\lambda\nu'} = \eta_{\lambda\nu'} \int \frac{d\omega d^2k}{(2\pi)^3} e^{-i\omega(t-t')+i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{x}'_\perp)} g_{D,N}(z,z'), \tag{3.29}$$

having defined

$$g_D(z,z';\kappa) \equiv \frac{\sin\kappa(az_<)\sin\kappa(a-z_>)}{\kappa\sin\kappa a}, \qquad 0<z,z'<a, \tag{3.30}$$

$$g_N(z,z';\kappa) \equiv -\frac{\cos\kappa(az_<)\cos\kappa(a-z_>)}{\kappa\sin\kappa a}, \qquad 0<z,z'<a, \tag{3.31}$$

where D, N stand for homogeneous Dirichlet or Neumann boundary conditions, respectively, $z_>$ $(z_<)$ are the larger (smaller) between z and z', while $\vec{k}_\perp$ has components (k_x,k_y), $\vec{x}_\perp$ has components (x,y), $\kappa \equiv \sqrt{\omega^2-k^2}$, and

$$G^{(1)}_{\mu\nu'} = \int \frac{d\omega d^2k}{(2\pi)^3} e^{-i\omega(t-t')+i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{x}'_\perp)} \Phi_{\mu\nu'}, \tag{3.32}$$

where the Φ components different from zero are written in Appendix B. A scalar field satisfies the same equations of the 22 component of the gauge field, hence we do not write it explicitly. In the following we will write simply $G_{\mu\nu'}$ and G for the Green function of the gauge and ghost field, respectively.

We should stress at this stage that, in general, the integrals defining the Green functions are divergent. They are well defined until $x \neq x'$, hence we will perform all our calculations maintaining the points separated and only in the very end shall we take the coincidence limit as $x' \to x$ [20]. We have decided to write the divergent terms explicitly so as to bear them in mind and remove them only in the final calculations by hand, instead of making the subtraction at an earlier stage.

Incidentally, we note that these Green functions satisfy the Ward identity (see Appendix A)

$$G^{\mu}_{\ \nu';\mu} + G_{;\nu'} = 0, \quad G^{\mu\ \ ;\nu'}_{\ \nu'} + G^{;\mu} = 0, \tag{3.33}$$

to first order in ϵ so that, to this order, gauge invariance is explicitly preserved (the check being simple but nontrivial).

IV. ENERGY-MOMENTUM TENSOR

By virtue of the formulae of Sec. III we get, from the asymptotic expansion $T_{\mu\nu'} \sim T^{(0)}_{\mu\nu'} + \frac{\epsilon}{a} T^{(1)}_{\mu\nu'} + \mathrm{O}(\epsilon^2)$ (here we present only the final results for brevity, while the complete calculation is reproduced in Appendix B),

$$\langle T^{(0)\mu\nu'} \rangle = \frac{1}{16\, a^4\, \pi^2} \left(\zeta_H \left(4, \frac{2\,a + z - z'}{2\,a} \right) + \zeta_H \left(4, \frac{z' - z}{2\,a} \right) \right) \mathrm{diag}(-1, 1, 1, -3), \tag{4.1}$$

where ζ_H is the Hurwitz ζ-function $\zeta_H(x, \beta) \equiv \sum_{n=0}^{\infty} (n + \beta)^{-x}$. On taking the limit $z' \to z^+$ we get

$$\lim_{z' \to z^+} \langle T^{(0)\mu\nu'} \rangle = \left(\frac{\pi^2}{720 a^4} + \lim_{z' \to z^+} \frac{1}{\pi^2 (z - z')^4} \right) \mathrm{diag}(-1, 1, 1, -3), \tag{4.2}$$

where the divergent term as $z' \to z$ can be removed by subtracting the contribution of infinite space without bounding surfaces [7], and in our analysis we therefore discard it hereafter. The renormalization of the energy-momentum tensor in curved spacetime is usually performed by subtracting the $\langle T_{\mu\nu} \rangle$ constructed with an Hadamard or Schwinger–DeWitt two-point function up to the fourth adiabatic order [1, 2]. In our problem, however, as we work to first order in ϵ, we are neglecting tidal forces and therefore the geometry of spacetime in between the plates is flat. Thus, we need only subtract the contribution to the energy momentum tensor that is independent of a, which is the standard subtraction in the context of the Casimir effect in flat spacetime.

In the same way (see Appendix B) we get, to first order in ϵ:

$$\begin{aligned} \lim_{z' \to z^+} \langle T^{(1)\mu\nu'} \rangle &= \mathrm{diag}(T^{(1)00}, T^{(1)11}, T^{(1)22}, T^{(1)33}) \\ &\quad + \lim_{z' \to z^+} \mathrm{diag}\Big(- z'/\pi^2 (z - z')^4, 0, 0, 0 \Big), \end{aligned} \tag{4.3}$$

where

$$T^{(1)00} = -\frac{\pi^2}{1200\, a^3} + \frac{\pi^2\, z}{3600\, a^4} + \frac{\pi\, \cot(\frac{\pi\, z}{a})\, \csc^2(\frac{\pi\, z}{a})}{240\, a^3}, \tag{4.4}$$

$$T^{(1)11} = \frac{\pi^2}{3600\, a^3} - \frac{\pi^2\, z}{1800\, a^4} - \frac{\pi\, \cot(\frac{\pi\, z}{a})\, \csc^2(\frac{\pi\, z}{a})}{120\, a^3}, \tag{4.5}$$

$$T^{(1)22} = T^{(1)11}, \tag{4.6}$$

$$T^{(1)33} = -\frac{(\pi^2\, (a - 2\, z))}{720\, a^4}. \tag{4.7}$$

By virtue of the Ward identities (3.33), here checked up to first order in ϵ, the gauge-breaking part of the energy-momentum tensor is found to be minus the ghost part, hence we compute the second only, and the result is written in Appendix B.

V. CASIMIR ENERGY AND FORCE

To compute the Casimir energy we must project the energy-momentum tensor along the unit timelike vector u with covariant components $u_\mu = (\sqrt{-g_{00}}, 0, 0, 0)$ to obtain $\rho = \langle T^{\mu\nu}\rangle u_\mu u_\nu$, so that

$$\begin{aligned}\rho \;&=\; \left(1+\epsilon\frac{z}{a}\right)\left[-\frac{\pi^2}{720a^4}+\frac{\epsilon}{a}\left(-\frac{\pi^2}{1200\,a^3}+\frac{\pi^2\,z}{3600\,a^4}+\frac{\pi\,\cot(\frac{\pi\,z}{a})\csc^2(\frac{\pi\,z}{a})}{240\,a^3}\right)\right]\\ &=\; -\frac{\pi^2}{720a^4}+2\frac{g}{c^2}\left(-\frac{\pi^2}{1200\,a^3}-\frac{\pi^2\,z}{900\,a^4}+\frac{\pi\,\cot(\frac{\pi\,z}{a})\csc^2(\frac{\pi\,z}{a})}{240\,a^3}\right)+\mathrm{O}(g^2),\end{aligned} \tag{5.1}$$

where in the second line we have substituted ϵ by its expression in terms of g. Thus, the energy stored in the Casimir device is found to be

$$E = \int d^3\Sigma\sqrt{-g}\langle T^{\mu\nu}\rangle u_\mu u_\nu = -\frac{\hbar c\pi^2}{720}\frac{A}{a^3}\left(1+\frac{5}{2}\frac{ga}{c^2}\right), \tag{5.2}$$

where A is the area of the plates, $d^3\Sigma$ is the three-volume element of an observer with four-velocity u_μ, Eq. (5.2) is expressed through a principal-value integral, and we have reintroduced $\hbar$ and c.

In the same way, the pressure on the plates is given by

$$P(z=0) = \frac{\pi^2}{240}\frac{\hbar c}{a^4}\left(1+\frac{2}{3}\frac{ga}{c^2}\right),\; P(z=a) = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}\left(1-\frac{2}{3}\frac{ga}{c^2}\right), \tag{5.3}$$

so that a net force pointing upwards along the z-axis is obtained, in full agreement with Eq. (8) in Ref. [12], with magnitude

$$F = \frac{\pi^2}{180}\frac{A\hbar g}{ca^3}. \tag{5.4}$$

The reader may wonder whether the pressure on the outer faces of the cavity may alter this result. A simple way to answer this question is to imagine that our cavity is included into a surrounding cavity on both sides. On denoting by b the common separation between either plates of the original cavity and the nearest plate of the surrounding cavity, and assuming that b is such that $a/b << 1$, but still small enough so as to obtain $gb/c^2 << 1$, we see from Sec. IV that the outer pressure on both plates of the original cavity includes the same divergent contribution which acts from within plus a finite contribution that becomes negligible for $a/b << 1$. To sum up, the divergent contributions to the pressure from the inside and the outside of either plate cancel each other exactly, and one is left just with the finite contribution from the inside, as given in Eq. (5.3).

As a check of the result, it can be verified that the energy-momentum tensor is covariantly conserved to first order in ϵ. To this order, the covariant conservation law implies the following conditions

$$\epsilon^0 : \ \langle T^{(0)\mu\nu} \rangle_{,\nu} = 0, \tag{5.5}$$

$$\epsilon : \ \begin{cases} \langle T^{(1)ij} \rangle_{,j} = 0 \quad (i = 0, 1, 2), \\ \\ \frac{1}{2} \left(\langle T^{(0)00} \rangle + \langle T^{(0)33} \rangle \right) + \langle T^{(1)33} \rangle_{,3} = 0, \end{cases} \tag{5.6}$$

that are indeed satisfied. Moreover, from the previous expressions of the energy-momentum tensor the following trace anomaly τ is obtained:

$$\tau = \frac{\hbar g}{c} \left(\frac{\pi^2 z}{180 a^4} - \frac{\pi}{24 a^3} \cot \left(\frac{\pi z}{a} \right) \csc^2 \left(\frac{\pi z}{a} \right) \right). \tag{5.7}$$

The volume integral of this density exists as a principal-value integral and is given by

$$\int \tau \ d^3\Sigma = \frac{\pi^2}{360} \frac{\hbar g}{c a^2} A. \tag{5.8}$$

The global, integrated form (5.8) of the trace anomaly Eq. (5.7) is the new result with respect to the analysis in Ref. [12]. It tends to zero at large separation a between the plates. This trace anomaly is therefore caused by the presence of the boundaries, and then is of a different nature from the usual trace anomaly which is encountered in curved spacetimes without boundaries, which depends on the Riemann curvature [17, 20].

VI. CONCLUDING REMARKS

The Casimir effect for scalar fields in curved spacetime [21] has been previously considered by various authors, in a number of different geometries [22]–[30]. More precisely, Refs. [22, 23] focus on a massless scalar field in half of the Einstein static universe, while Ref. [24] studies the same field in a Friedmann background geometry with spherical boundary, and the associated nonintegrable divergence in the renormalized energy density. Massless scalar fields in spherical or cylindrical shells are studied also in Refs. [25]–[29], with local boundary conditions of the Dirichlet or Robin type. Moreover, the work in Ref. [30] deals with the more complicated problem of a massive nonminimally coupled scalar field in between two infinite parallel plates moving by uniform proper acceleration. Such a scalar field is taken to obey Robin boundary conditions on the plates, and the interaction forces between the plates are investigated as functions of the proper accelerations and coefficients in the boundary conditions. Interestingly, for some values of these parameters the interaction forces are found to be repulsive at small distances and attractive at large distances [30].

To the best of our knowledge, the analysis presented in this paper represents the first study of the energy-momentum tensor for the electromagnetic field in a Casimir cavity placed in a weak gravitational field. The resulting calculations are considerably harder than in the case of scalar fields. By using Green-function techniques, we have evaluated the influence of the gravity acceleration on the regularized energy-momentum tensor of the quantized electromagnetic field between two plane-parallel ideal metallic plates, at rest in the gravitational field of the earth, and lying in a horizontal plane. In particular, we have obtained a detailed derivation of the theoretical prediction according to which a Casimir device in a weak gravitational field will experience a tiny push in the upwards direction [12]. This result is consistent with the picture that the *negative* Casimir energy in a gravitational field will behave like a *negative mass*. Furthermore, we find a trace anomaly in Eq. (5.8) proportional to the gravitational acceleration and vanishing for infinite plates' separation, not previously worked out for a Casimir device in a gravitational field. Our original results are relevant both for quantum field theory in curved space-time, and for the theoretical investigation of vacuum energy effects (see below).

We stress that in our computation we do not add by hand a mass term for photons, unlike the work in Ref. [2], since this regularization procedure breaks gauge invariance even prior to adding a gauge-fixing term, and is therefore neither fundamental nor desirable [6, 20]. In agreement with the findings of Deutsch and Candelas for conformally invariant fields [4], we find that on approaching either wall, the energy density of the electromagnetic field diverges as the third inverse power of the distance from the wall. It is interesting to point out that, in the intermediate stages of the computation, quartic divergences appear in the contributions from the ghost and the gauge breaking terms, which however cancel each other exactly. The occurrence of these higher divergences in such terms is also consistent with the results of Deutsch and Candelas, in view of the obvious fact that ghost fields are not ruled by conformally invariant operators. Unfortunately, a quantitative comparison with their results is not possible because they *assume* a traceless tensor, which is not the case in our problem where a trace anomaly is found to arise.

Our results, jointly with the work in Refs. [12, 31], are part of a research program aimed at studying the Casimir effect in a weak gravitational field, with possible corrections (albeit small) to the attractive force on the plates resulting from spacetime curvature [32] (cf. the recent theoretical analysis of quantum vacuum engineering propulsion in Ref. [33]). Hopefully, these efforts represent a first step towards an experimental verification of the validity of the Equivalence Principle for virtual photons.

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APPENDIX A: ON WARD IDENTITIES AND BOUNDARY CONDITIONS FOR GAUGE FIELDS

In the case of gauge theories there exists, on the space Φ of field histories, a set of vector fields Q_α that leave the action S invariant, i.e. [17]

$$Q_\alpha S = 0. \tag{A1}$$

On denoting by ${Q^i}_\alpha$ the components of such vector fields, and by $S_{,i}$ the functional derivatives of the action with respect to field variables φ^i, Eq. (A1) implies that

$$S_{,i}{Q^i}_\alpha = 0. \tag{A2}$$

By virtue of Eq. (A2), the operator $S_{,ij}$ is not invertible, and an invertible operator U_{ij} is obtained upon adding to S the gauge-breaking term

$$\frac{1}{2}P^\alpha \omega_{\alpha\beta} P^\beta = \frac{1}{2}\int d^4x \int d^4x' P^\alpha(x)\omega_{\alpha\beta}(x,x')P^\beta(x'), \tag{A3}$$

where P^α is the gauge-fixing functional and $\omega_{\alpha\beta}(x,x') = \omega_{\alpha\beta'}$ is a nonsingular, symmetric continuous matrix, possibly depending on field variables to achieve full covariance of the formalism [34]. The desired invertible gauge-field operator U_{ij} reads therefore

$$U_{ij} \equiv S_{,ij} + P^\alpha_{\ ,i}\, \omega_{\alpha\beta}\, P^\beta_{\ ,j}, \tag{A4}$$

which should be considered jointly with the ghost operator

$$\widehat{U}^{\alpha}_{\ \beta} \equiv Q_{\beta} P^{\alpha} = P^{\alpha}_{\ ,i} \, Q^{i}_{\ \beta}, \tag{A5}$$

since the full action in the functional integral for the $\langle \text{out}|\text{in} \rangle$ amplitude is given by

$$S + \frac{1}{2} P^{\alpha} \omega_{\alpha\beta} P^{\beta} + \chi_{\alpha} \widehat{U}^{\alpha}_{\ \beta} \psi^{\beta},$$

with χ_{α} and ψ^{β} independent ghost fields [17], which obey Fermi statistics if the gauge fields φ^{i} are bosonic.

Repeated functional differentiation of Eq. (A2) yields the classical Ward identities of the theory, which can be used to derive remarkable identities among the gauge-field and ghost Green functions. For example, the first functional derivative of Eq. (A2) yields

$$S_{,ij} Q^{i}_{\ \alpha} + S_{,i} Q^{i}_{\ \alpha,j} = 0. \tag{A6}$$

We now re-express $S_{,ij}$ from Eq. (A4) and bear in mind the definition (A5) to find

$$U_{ij} Q^{j}_{\ \alpha} - \widehat{U}^{\beta}_{\ \alpha} \, \omega_{\beta\delta} \, P^{\delta}_{\ ,j} + S_{,i} Q^{i}_{\ \alpha,j} = 0. \tag{A7}$$

Restriction to the dynamical subspace, where $S_{,i}$ vanishes, and composition with the ghost Green function $\widehat{G}^{\alpha\gamma}$, for which

$$\widehat{U}^{\beta}_{\ \alpha} \, \widehat{G}^{\alpha\gamma} = -\delta^{\beta\gamma}, \tag{A8}$$

yields

$$U_{ij} Q^{i}_{\ \alpha} \, \widehat{G}^{\alpha\gamma} + \omega^{\gamma}_{\ \delta} \, P^{\delta}_{\ ,j} = 0. \tag{A9}$$

Now we act on Eq. (A9) with the gauge-field Green function G^{jk} (see Eq. (3.1)), finding therefore [34]

$$Q^{k}_{\ \alpha} \, \widehat{G}^{\alpha\gamma} = \omega^{\gamma}_{\ \delta} \, P^{\delta}_{\ ,j} \, G^{jk}. \tag{A10}$$

This equation holds for all type-I theories, i.e. all gauge theories for which the Lie bracket of the vector fields in Eq. (A1) is a linear combination of such fields with structure constants, i.e.

$$[Q_{\alpha}, Q_{\beta}] = C^{\gamma}_{\ \alpha\beta} \, Q_{\gamma}, \tag{A11}$$

with $C^{\gamma}_{\ \alpha\beta,i} = 0$. For the case of Maxwell theory in curved spacetime, the ghost Green function has no explicit group indices, $Q^{k}_{\ \alpha}$ reduces to covariant derivatives with respect to the Levi–Civita connection, and $\omega_{\ \ \delta}^{\gamma} \ P^{\delta}_{\ ,j} G^{jk}$ yields the covariant derivative of the photon Green function. Thus, the Ward identity (3.33) is eventually obtained. For the sake of completness we write down the expansion to first order in ϵ of the first of Eqs. (3.33):

$$\epsilon^0) \ \ \eta^{\mu\nu} H^{(0)}_{\mu\alpha',\nu} + H^{(0)}_{,\alpha'} = 0 \quad \Rightarrow \quad \begin{cases} -H^{(0)}_{00',0} + H^{(0)}_{,0'} = 0, \\ H^{(0)}_{ii',i} + H^{(0)}_{,i'} = 0, \end{cases} \tag{A12}$$

$$\epsilon) \ \ \eta^{\mu\nu} H^{(1)}_{\mu\alpha',\nu} - a\eta^{\mu\nu} \frac{\Gamma^{\rho}_{\mu\nu}}{\epsilon} H^{(0)}_{\rho\alpha'} + z\delta^{\mu}_0 \delta^{\nu}_0 H^{(0)}_{\mu\alpha',\nu} + H^{(1)}_{,\alpha'} = 0 \quad \Rightarrow$$

$$\begin{cases} -H^{(1)}_{00',0} - H^{(1)}_{30',3} + zH^{(0)}_{00',0} + H^{(1)}_{,0'} = 0, \\ \\ \qquad H^{(1)}_{ii',i} + H^{(0)}_{,i'} = 0 \ \ (i = 1, 2), \\ \\ -H^{(1)}_{03',0} - H^{(1)}_{33',3} + \frac{1}{2} H^{(0)}_{33'} + H^{(1)}_{,3'} = 0. \end{cases} \tag{A13}$$

A recipe for writing gauge-invariant boundary conditions [35] in field theory is expressed by the equations

$$\Bigr[\pi^{i}_{\ j} \ \varphi^{j}\Bigr]_{\partial M} = 0, \tag{A14}$$

$$\Bigr[P^{\alpha}[\varphi]\Bigr]_{\partial M} = 0, \tag{A15}$$

$$[\psi_{\beta}]_{\partial M} = 0, \tag{A16}$$

where $\pi^{i}_{\ j}$ is a tangential projection operator. In the case of Maxwell theory, considered in our paper, Eqs. (A14)–(A16) become

$$[A_k]_{\partial M} = 0, \tag{A17}$$

$$[\Phi(A)]_{\partial M} = 0, \tag{A18}$$

$$[\lambda]_{\partial M} = 0. \tag{A19}$$

On performing the familiar gauge transformation

$${}^{\lambda}A_{\mu} \equiv A_{\mu} + \nabla_{\mu}\lambda, \tag{A20}$$

Eq. (A17) is preserved under (A20) if and only if λ vanishes on the boundary (see (A19)), since tangential derivatives and restriction to the boundary are commuting operations [35]. Moreover, under (A20), the gauge-fixing functional changes according to

$$\Phi(A) - \Phi({}^{\lambda}A) = \widehat{U} \ \lambda, \tag{A21}$$

where $\widehat{U}$ is a linear differential operator. Now if λ is expanded according to a complete orthonormal set of eigenfunctions of $\widehat{U}$, i.e.

$$\widehat{U}\, u_i = \mu_i\, u_i, \tag{A22}$$

$$\lambda = \sum_i C_i\, u_i, \tag{A23}$$

the gauge invariance of the boundary condition (A18) is again guaranteed by (A19), because

$$\Phi(A) - \Phi({}^{\lambda}A) = \widehat{U}\, \lambda = \sum_i C_i\, \mu_i\, u_i. \tag{A24}$$

The vanishing of u_i on the boundary implies therefore that both λ and $\Phi({}^{\lambda}A)$ vanish therein, if $\Phi(A)$ was already satisfying Eq. (A18). In the full quantum theory, λ should be replaced by two fermionic ghost fields [17, 19].

APPENDIX B: GREEN FUNCTIONS AND ENERGY-MOMENTUM TENSORS

1. The Fourier transform of $G^{(1)}_{\lambda\nu'}$

In the following, for the sake of simplicity, we always assume $z' \geq z$, hence we have to be careful when computing integrals and limits, but the resulting formulae become relatively less cumbersome. With the notation in Eq. (3.32) we find (hereafter for the sake of brevity we define $\xi \equiv a\kappa,\ s(\xi) \equiv z\kappa = \frac{z\xi}{a}$ and $s'(\xi) \equiv z'\kappa = \frac{z'\xi}{a}$)

$$
\begin{aligned}
\Phi^{00'} &= \frac{a\sin^{-2}(\xi)}{8\xi^4}\Bigg[-a^2\omega^2\cos(2\xi-s-s')s^2 + a^2\omega^2\cos(s)\cos(s')s^2 \\
&+ \sin(s)\Bigg(a^2\omega^2\left((s^2-s'^2)\sin(2\xi-s') + \left(2\xi^2-s'^2\right)\sin(s')\right) \\
&- 2\left(\xi^2+a^2\omega^2\right)(s+s')\sin(\xi)\sin(\xi-s')\Bigg)\Bigg], && \text{(B1)} \\
\Phi^{03'} &= -\frac{ia^2}{2\xi^3}\omega\sin^{-1}(\xi)\sin(s)((s-s')\cos(\xi-s')-\sin(\xi-s')), && \text{(B2)} \\
\Phi^{11'} &= \frac{a\sin^{-2}(\xi)}{8\xi^4}\Bigg[a^2\omega^2\cos(2\xi-s-s')s^2 - a^2\omega^2\cos(s)\cos(s')s^2 \\
&+ \sin(s)\Bigg(a^2\omega^2\left(\left(s'^2-s^2\right)\sin(2\xi-s') + \left(s'^2-2\xi^2\right)\sin(s')\right) \\
&- 2(\xi^2-a^2\omega^2)(s+s')\sin(\xi)\sin(\xi-s')\Bigg)\Bigg], && \text{(B3)} \\
\Phi^{22'} &= \Phi^{11'}, && \text{(B4)} \\
\Phi^{30'} &= -\frac{ia^2\omega\sin^{-1}(\xi)}{2\xi^3}\Bigg((s'-s)\cos(s)+\sin(s)\Bigg)\sin(\xi-s'), && \text{(B5)} \\
\Phi^{33'} &= \frac{a\sin^{-1}(\xi)}{8\xi^4}\Bigg[2\left(\xi^2-a^2\omega^2\left(s^2-1\right)\right)\cos(\xi-s')\sin(s) \\
&+ \cos(s)\Bigg(\csc(\xi)\left(\left(\xi^2-a^2\omega^2\left(s'^2-1\right)\right)\cos(2\xi-s')\right. \\
&+ \left.\cos(s')\left(-\left(2a^2\omega^2+1\right)\xi^2 + a^2\omega^2\left(s'^2-1\right) + \left(\xi^2-a^2\omega^2\right)(s+s')\sin(2\xi)\right)\right) \\
&+ 2\left(\xi^2-a^2\omega^2\right)(s+s')\sin(\xi)\sin(s')\Bigg)\Bigg]. && \text{(B6)}
\end{aligned}
$$

2. The energy-momentum tensors

In our analysis we deal with

$$\langle T^{(0)\mu\nu'}\rangle = 2i\int \frac{d\omega d^2k}{(2\pi)^3} e^{-i\omega(t-t')+i\vec{k}_\perp\cdot(\vec{x}-\vec{x}')_\perp}\widetilde{T}^{(0)\mu\nu'}[\omega,\vec{k};z,z'], \tag{B7}$$

where the $2i$ factor results from the relation (3.3) between the Hadamard and Feynman Green functions, and $\widetilde{T}^{(0)\mu\nu'}$ is a symmetric tensor whose components are

$$\begin{aligned}
\widetilde{T}^{(0)00'} &= -\frac{a\,\omega^2}{2\xi}\cos(\xi+s-s')\csc(\xi),\\
\widetilde{T}^{(0)01'} &= -\frac{a\,k_x\,\omega}{2\xi}\cos(\xi+s-s')\csc(\xi),\\
\widetilde{T}^{(0)02'} &= -\frac{a\,k_y\,\omega}{2\xi}\cos(\xi+s-s')\csc(\xi),\\
\widetilde{T}^{(0)03'} &= -\frac{i}{2}\,\omega\csc(\xi)\sin(\xi+s-s'),\\
\widetilde{T}^{(0)11'} &= \frac{\xi^2+a^2\left(k_y^2-\omega^2\right)}{2\,a\,\xi}\cos(\xi+s-s')\csc(\xi),\\
\widetilde{T}^{(0)12'} &= -\frac{a\,k_x\,k_y}{2\xi}\cos(\xi+s-s')\csc(\xi),\\
\widetilde{T}^{(0)13'} &= -\frac{i}{2}\,k_x\sin(\xi+s-s')\csc(\xi),\\
\widetilde{T}^{(0)22'} &= -\frac{a\,{k_y}^2}{2\xi}\cos(\xi+s-s')\csc(\xi),\\
\widetilde{T}^{(0)23'} &= -\frac{i}{2}\,k_y\;\sin(\xi+s-s')\csc(\xi),\\
\widetilde{T}^{(0)33'} &= -\frac{\xi}{2\,a}\cos(\xi+s-s')\csc(\xi).
\end{aligned}$$

Now we first take the limit as $t'\to t$ and $\vec{x}'_\perp\to\vec{x}_\perp$. At that stage, on taking $z\neq z'$ we pass the limit under the integral, then sending $\omega\to i\omega$, $(\kappa\to i\kappa)$, and going to spherical coordinates: $\omega^2\to-\xi^2\,\cos^2\theta$, $k_y^2\to\xi^2\,\sin^2\theta\,\sin^2\phi$, angular integration yields eventually

$$\langle T^{(0)\mu\nu'}\rangle[z,z'] = \frac{1}{6\,\pi^2}\frac{1}{a^4}\int d\xi\;\xi^3\times\,\cosh(\xi-s-s')\,\mathrm{csch}(\xi)\mathrm{diag}(-1,1,1,-3). \qquad \text{(B8)}$$

After integrating over the ξ variables we obtain Eq. (4.1). Analogously, upon integrating over the solid angle we get, to first order in momentum space,

$$
\begin{aligned}
\langle T^{(1)00'}\rangle[z,z'] &= \frac{1}{120\pi^2}\int d\xi\, \xi^2 \sinh^{-1}(\xi)\Bigg[60\,(-s+s')\,\cosh(\xi-s-s') \\
&+ 4\,(3\,s-2\,s')\,\cosh(\xi+s-s')+3\,\left(s^2-s'^2\right)\,\sinh(\xi+s-s') \\
&+ \left(\cosh(2\,\xi-s-s')-3\,\xi^2\,\cosh(s-s')-\cosh(s+s')\right)\sinh^{-1}(\xi)\Bigg], \qquad \text{(B9)} \\
\langle T^{(1)11'}\rangle[z,z'] &= \frac{1}{120\pi^2}\int d\xi\xi^2 \sinh^{-1}(\xi)\Bigg[-4\,(s+s')\,\cosh(\xi+s-s')+\Big(-2\,\cosh(2\,\xi-s-s') \\
&+ \xi^2\,\cosh(s-s')+2\,\cosh(s+s')\Big)\sinh^{-1}(\xi) \\
&+ \left(-s^2+s'^2\right)\,\sinh(\xi+s-s')\Bigg], \qquad \text{(B10)} \\
\langle T^{(1)22'}\rangle[z,z'] &= T^{(1)11'}, \qquad \text{(B11)} \\
\langle T^{(1)33'}\rangle[z,z'] &= \frac{1}{24\pi^2}\int d\xi\xi^2 \sinh^{-1}(\xi)\Bigg[-\left(\xi^2\,\cosh(s-s')\sinh^{-1}(\xi)\right) \\
&+ (s+s')\,\Big(4\,\cosh(\xi+s-s')+(s-s')\,\sinh(\xi+s-s')\Big)\Bigg], \qquad \text{(B12)}
\end{aligned}
$$

where we have written explicitly only nonvanishing components. Integration over ξ finally yields Eqs. (4.4)–(4.7).

For completeness we write also the ghost energy-momentum tensor $\langle T^{\mu\nu}\rangle_{\rm gh}$ for Dirichlet boundary conditions, i.e.

$$
\begin{aligned}
\langle T^{(0)00}\rangle_{\rm gh} &= \frac{\pi^2}{720\,a^4}+\frac{\pi^2\,\left(2+\cos(\frac{2\,\pi\,z}{a})\right)\,\csc^4(\frac{\pi\,z}{a})}{24\,a^4}+\lim_{z'->z^+}\frac{1}{\pi^2\,(z-z')^4}, \qquad \text{(B13)} \\
\langle T^{(0)11}\rangle_{\rm gh} &= -\frac{\pi^2}{720\,a^4}-\frac{\pi^2\,\left(2+\cos(\frac{2\,\pi\,z}{a})\right)\,\csc^4(\frac{\pi\,z}{a})}{24\,a^4}-\lim_{z'->z^+}\frac{1}{\pi^2\,(z-z')^4}, \qquad \text{(B14)} \\
\langle T^{(0)22}\rangle_{\rm gh} &= \langle T^{(0)11}\rangle_{\rm gh}, \qquad \text{(B15)} \\
\langle T^{(0)33}\rangle_{\rm gh} &= \frac{\pi^2}{240a^4}+\lim_{z'->z^+}\frac{3}{\pi^2(z-z')^4}, \qquad \text{(B16)}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle T^{(1)00}\rangle_{\rm gh} &= \frac{\pi^2\ (3\,a-11\,z)}{3600\,a^4}+\frac{\pi^2}{40\,a^5}\ \Big(a\ (a-7\,z)-\pi z(a-z)\cot(\frac{\pi\,z}{a})\Big)\ \csc^4(\frac{\pi\,z}{a})\\
&+\ \frac{\pi}{240\,a^5}\ \csc^2(\frac{\pi\,z}{a})\ \Big((5\,a^2+2\pi^2 z(a-z))\cot(\frac{\pi\,z}{a})-4a\pi(a-7z)\Big)\\
&-\ \lim_{z'->z^+}\frac{z}{\pi^2\,(z-z')^4}, \qquad\qquad \text{(B17)}\\
\langle T^{(1)11}\rangle_{\rm gh} &= -\frac{(\pi^2\ (a-2\,z))}{3600\,a^4}\\
&-\ \frac{\pi^2}{160\,a^6}\ \Big(a(8a^2-15az-9z^2)-8\pi z(a^2-az-z^2)\cot(\frac{\pi\,z}{a})\Big)\ \csc^4(\frac{\pi\,z}{a})\\
&-\ \frac{\pi}{480\,a^6}\ \Bigg((10a^3+a^2(3+8\pi^2)z-8\,\pi^2 z^2(a+z))\cot(\frac{\pi\,z}{a})\\
&+\ 2a\pi(-8a^2+15az+9z^2)\Bigg)\csc^2(\frac{\pi\,z}{a}), \qquad\qquad \text{(B18)}\\
\langle T^{(1)22}\rangle_{\rm gh} &= \langle T^{(1)11}\rangle_{\rm gh}, \qquad\qquad \text{(B19)}\\
\langle T^{(1)33}\rangle_{\rm gh} &= \frac{\pi^2(a-2z)}{720\,a^4}-\frac{\pi\csc^2(\frac{\pi\,z}{a})}{96\,a^4}\cot(\frac{\pi\,z}{a})\\
&-\ \frac{1}{64\pi^2\,a^4}\ \Big(4a\Big(\psi^{(1)}(1/2-z/a)-\psi^{(1)}(-1/2+z/a)\Big)\\
&+\ (a-2z)\Big(\psi^{(2)}(1/2-z/a)+\psi^{(2)}(-1/2+z/a)\Big)\Big), \qquad\qquad \text{(B20)}
\end{aligned}
$$

where $\psi^{(n)}(z)$ is the n-th derivative of the logarithmic derivative $\psi(z)$ of the Γ-function.

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